Claude Opus 4.7

Setup. Define $A := \lceil n/a_w \rceil$ and $B := \lfloor a_w/r \rfloor$. For any $b \in \mathbb{N}$:

$$(a_w, b) \in W \iff \tfrac{n}{a_w} \le b \le \tfrac{a_w}{r} \iff A \le b \le B.$$

So the valid b's form the integer interval $[A, B]$. We must show $A = B$.

Assume for contradiction $A < B$, i.e., $A \le B - 1$. We'll derive $a_w \ge rb_h$, contradicting the hypothesis $a_w < rb_h$.

Step 1. $a_w(B-1) \ge n$.

Since $B - 1 \ge A \ge n/a_w$, multiply by a_w.

Step 2. $rB \le a_w < rB + 1$.

The lower bound is by definition of B. For the upper bound, suppose $a_w \ge rB + 1$. Then $a_w - 1 \ge rB$, while $(a_w - 1)/r < a_w/r < B + 1$, forcing $\lfloor (a_w - 1)/r \rfloor = B$.

Minimality of a_w means $(a_w - 1, b) \notin W$ for any b, so

$$\lceil n/(a_w - 1) \rceil > \lfloor (a_w - 1)/r \rfloor = B \implies n > B(a_w - 1).$$

Combined with Step 1:

$$B(a_w - 1) < n \le a_w(B - 1) \implies a_w B - B < a_w B - a_w \implies a_w < B.$$

But $r \ge 1$ gives $a_w \ge rB \ge B$. Contradiction.

Step 3. $\lceil n/B \rceil \le a_w - 1$.

By Step 1, $n/B \le a_w(B-1)/B = a_w - a_w/B$. Since $a_w/B \ge r \ge 1$, we get $a_w - a_w/B \le a_w - 1$. As $a_w - 1 \in \mathbb{Z}$, $\lceil n/B \rceil \le a_w - 1$.

Step 4. $\lfloor rB \rfloor \ge a_w - 1$.

By Step 2, $rB > a_w - 1$. Since $a_w - 1$ is an integer below rB, $\lfloor rB \rfloor \ge a_w - 1$.

Step 5. Combining Steps 3 and 4:

$$\lceil n/B \rceil \;\le\; a_w - 1 \;\le\; \lfloor rB \rfloor.$$

So pick any integer $a^* \in [\lceil n/B \rceil, \lfloor rB \rfloor]$. Then $a^* B \ge n$ and $a^* \le rB$, so $(a^*, B) \in H$. Therefore $b_h \le B$.

Conclusion. $rb_h \le rB \le a_w$, contradicting $a_w < rb_h$.

Hence $A = B$, and b_w is the unique b with $(a_w, b) \in W$. ■

The geometric picture, for intuition: W sits above the hyperbola $ab = n$ and below the line $b = a/r$; H is above both. In the continuous world the minima coincide at $(\sqrt{rn}, \sqrt{n/r})$. The hypothesis $a_w < rb_h$ says the discrete corner of W sits *strictly* to the left of $r \cdot b_h$ — and the proof shows this margin is too tight for two integers to fit between n/a_w and a_w/r. If two did, you could squeeze a point of H at $b = B < b_h$, breaking b_h's minimality.