DeepSeek v4

We are given $r \geq 1$ (a real number), a positive integer n, and the sets

$$W = \{(a,b) \in \mathbb{N}^2 \mid n \leq ab,\ a \geq rb\}, \qquad H = \{(a,b) \in \mathbb{N}^2 \mid n \leq ab,\ rb \geq a\}.$$

Let (a_w, b_w) be a pair in W that minimises a, and let (a_h, b_h) be a pair in H that minimises b. Assume that

$$a_w < rb_h.$$

We must prove that no other integer $b \neq b_w$ satisfies $(a_w, b) \in W$.

Proof by contradiction.
Suppose there exists $b' \neq b_w$ with $(a_w, b') \in W$. Set $A = a_w$ and let

$$S = \{b \in \mathbb{N} \mid (A,b) \in W\} = \{b \in \mathbb{N} \mid Ab \geq n,\ A \geq rb\}.$$

By assumption $b_w, b' \in S$ and $b_w \neq b'$, so $|S| \geq 2$.
Let $b_{\max} = \max S$ and $b_{\min} = \min S$. Because the conditions are $b \geq \lceil n/A \rceil$ and $b \leq \lfloor A/r \rfloor$, we actually have

$$b_{\min} = \lceil n/A \rceil, \qquad b_{\max} = \lfloor A/r \rfloor.$$

Indeed, if $b_{\max} < \lfloor A/r \rfloor$ then $b_{\max} + 1 \leq \lfloor A/r \rfloor \leq A/r$ gives $A \geq r(b_{\max}+1)$ and $A(b_{\max}+1) = Ab_{\max} + A > n$, so $b_{\max}+1 \in S$, contradicting maximality. The argument for $b_{\min}$ is similar.

Since $|S| \geq 2$, we have $b_{\min} \leq b_{\max} - 1$.

1. The position of b_h.
By definition, b_h is the smallest b for which there exists an integer a with $a \leq rb$ and $ab \geq n$. This is equivalent to

$$b_h = \min\{b \in \mathbb{N} \mid \lceil n/b \rceil \leq rb\}.$$

For every $b < b_h$ we therefore have $\lceil n/b \rceil > rb$.

2. Relating A, $b_{\max}$ and b_h.
Because $b_{\max} \in S$, we have $A \geq rb_{\max}$. If $b_{\max} \geq b_h$, then $A \geq rb_{\max} \geq rb_h$, contradicting $A < rb_h$. Hence

$$b_{\max} < b_h.$$

By the property of b_h, we obtain $\lceil n/b_{\max} \rceil > rb_{\max}$.

Now $Ab_{\max} \geq n$ implies $A \geq \lceil n/b_{\max} \rceil$. If the inequality were strict ($A > \lceil n/b_{\max} \rceil$), then $A - 1 \geq \lceil n/b_{\max} \rceil > rb_{\max}$, so $(A-1)b_{\max} \geq n$ and $A - 1 \geq rb_{\max}$, making $(A-1, b_{\max}) \in W$ and contradicting the minimality of A in W. Therefore

$$A = \lceil n/b_{\max} \rceil \quad \text{and} \quad A > rb_{\max}.$$

From $b_{\max} = \lfloor A/r \rfloor$ we also have $A < r(b_{\max} + 1)$.
If $b_h > b_{\max} + 1$, then $b_{\max} + 1 < b_h$ implies $\lceil n/(b_{\max}+1) \rceil > r(b_{\max}+1)$. But $\lceil n/(b_{\max}+1) \rceil \leq \lceil n/b_{\max} \rceil = A$, giving $A > r(b_{\max}+1)$, contradicting $A < r(b_{\max}+1)$. Since we already know $b_h > b_{\max}$, we must have

$$b_h = b_{\max} + 1.$$

3. Deriving a contradiction.
Because $|S| \geq 2$, we have $b_{\min} \leq b_{\max} - 1$, so $b_{\max} - 1 \in S$. Hence $A(b_{\max} - 1) \geq n$, which gives $A \geq \lceil n/(b_{\max} - 1) \rceil$. The ceiling function is non-increasing, so $\lceil n/(b_{\max} - 1) \rceil \geq \lceil n/b_{\max} \rceil = A$. Consequently,

$$\lceil n/(b_{\max} - 1) \rceil = A.$$

Thus we simultaneously have

$$\lceil n/b_{\max} \rceil = A \quad \text{and} \quad \lceil n/(b_{\max} - 1) \rceil = A.$$

By definition of the ceiling function, these equalities mean

$$A - 1 < \frac{n}{b_{\max}} \leq A \qquad \text{and} \qquad A - 1 < \frac{n}{b_{\max} - 1} \leq A.$$

In particular,

$$n > b_{\max}(A-1) \quad \text{and} \quad n \leq (b_{\max} - 1)A.$$

For such an integer n to exist we need $b_{\max}(A-1) < (b_{\max} - 1)A$, which simplifies to $b_{\max}A - b_{\max} < b_{\max}A - A$, i.e. $A < b_{\max}$.

On the other hand, from step 2 we have $A > rb_{\max}$. Because $r \geq 1$, this yields $A > rb_{\max} \geq b_{\max}$, so $A > b_{\max}$.
The two inequalities $A < b_{\max}$ and $A > b_{\max}$ cannot hold simultaneously. This contradiction shows that our initial supposition was false.

Conclusion.

No integer $b \neq b_w$ satisfies $(a_w, b) \in W$. The point (a_w, b_w) is the unique element of W with first coordinate a_w.