Gemini 3.1 Pro

Let $W(a) = \{b \in \mathbb{N} \mid n \leq ab \text{ and } a \geq rb\}$.

By definition, a_w is the minimum a such that $W(a)$ is non-empty, and $b_w \in W(a_w)$ We want to prove that $|W(a_w)| = 1$.

Step 1: Assume Multiple Valid b's

Suppose for the sake of contradiction that there exists another $b \neq b_w$ such that $(a_w, b) \in W$. This means $|W(a_w)| \geq 2$.

Since the condition $a_w \geq rb$ implies $b \leq \lfloor a_w/r \rfloor$, the set $W(a_w)$ is bounded above. Let y be the maximum element in $W(a_w)$, so $y = \lfloor a_w/r \rfloor$.

Because we assumed $|W(a_w)| \geq 2$, the integer just below y must also be in $W(a_w)$. Let $b = y - 1$, which means $y = b + 1$.

Since both b and y are in $W(a_w)$, we have:

1. $a_w b \geq n$
2. $a_w y \geq n$ (and $a_w \geq ry$)

Step 2: Exploit the Bounding Condition on H

We are given the strict inequality assumption: $a_w < rb_h$.

Dividing by r, we get $a_w/r < b_h$. Since $y = \lfloor a_w/r \rfloor \leq a_w/r$, we establish that $y < b_h$.

Recall the definition of b_h: it is the minimum b-coordinate in the set $H = \{(a, b) \in \mathbb{N}^2 \mid n \leq ab \text{ and } rb \geq a\}$.

Because $y < b_h$, y cannot be the second coordinate of any element in H. This means there exists no integer a such that $ay \geq n$ and $ry \geq a$.

Equivalently, for all integers $a \leq ry$, we must have $ay < n$.

Let $A = \lfloor ry \rfloor$. Since A is an integer and $A \leq ry$, it immediately follows that:

$$Ay < n$$

Step 3: Construct a Competing Candidate for W

Consider the integer $a' = A + 1$. Let's test if $(a', y) \in W$.

First, we check the bounding condition $a' \geq ry$. Since $A = \lfloor ry \rfloor$, we know strictly that $ry < A + 1 = a'$. Thus, $a' \geq ry$ is satisfied.

Next, we must check if $a'y \geq n$. Let's assume for a moment that it isn't, meaning $a'y < n$.

Since n and $a'y$ are integers, $a'y < n \implies (A + 1)y \leq n - 1$.

We established in Step 1 that $a_w b \geq n$. Substituting this in, we get:

$$(A + 1)y \leq a_w b - 1$$

Substitute $y = b + 1$ to get everything in terms of b:

$$\begin{gathered}(A + 1)(b + 1) \leq a_w b - 1 \\ Ab + A + b + 2 \leq a_w \\ a_w \geq A + 1 + \frac{A + 2}{b}\end{gathered}$$

Now, let's look at the upper bound of a_w. Since $y = \lfloor a_w / r \rfloor$, we know $a_w / r < y + 1 = b + 2$, which gives $a_w < r(b + 2)$.

Additionally, $A = \lfloor ry \rfloor = \lfloor r(b + 1) \rfloor$, which implies $r(b + 1) < A + 1$, yielding $r < \frac{A+1}{b+1}$.

Combine these to bound a_w:

$$a_w < r(b + 2) < \frac{A + 1}{b + 1}(b + 2) = A + 1 + \frac{A + 1}{b + 1}$$

Let's chain our lower and upper bounds for a_w:

$$\begin{gathered}A + 1 + \frac{A + 2}{b} \leq a_w < A + 1 + \frac{A + 1}{b + 1} \\ \frac{A + 2}{b} < \frac{A + 1}{b + 1} \\ (A + 2)(b + 1) < b(A + 1) \\ Ab + A + 2b + 2 < Ab + b \\ A + b + 2 < 0\end{gathered}$$

Since $A \geq 1$ and $b \geq 1$, it is impossible for their sum plus two to be strictly less than zero. This is a contradiction. Therefore, our assumption that $a'y < n$ must be false. We conclude that:

$$a'y \geq n$$

Step 4: The Final Contradiction

Because $a' \geq ry$ and $a'y \geq n$, the pair (a', y) is a valid element of W.

Since a_w is defined as the absolute minimum a-coordinate in W, we must have $a_w \leq a'$.

Substituting $a' = A + 1$, we get $a_w \leq A + 1$.

However, from Step 2, we know that $Ay < n$, and from Step 1, $n \leq a_w b$. Therefore:

$$Ay < a_w b$$

Substitute $y = b + 1$ and $a_w \leq A + 1$:

$$\begin{gathered} A(b+1) < (A+1)b \\ Ab + A < Ab + b \\ A < b \end{gathered}$$

Let's evaluate A. We defined $A = \lfloor r(b+1) \rfloor$. Because $r \in [1, \infty)$, we know $r \geq 1$.

Therefore, $A \geq \lfloor 1 \cdot (b+1) \rfloor = b + 1$.

We have rigorously arrived at:

$$b + 1 \leq A < b \implies 1 < 0$$

This is an absolute mathematical contradiction.

Conclusion

The contradiction stems entirely from the assumption that $|W(a_w)| \geq 2$. Therefore, the set $W(a_w)$ can contain no more than a single element. If $(a_w, b) \in W$, then b must be exactly b_w. ■