# GPT 5.5 Pro

Assume, for contradiction, that there is some $b \neq b_w$ such that

$$(a_w, b) \in W.$$

Let

$$B = \max(b, b_w), \qquad C = \min(b, b_w).$$

Then $B > C$, so since $B, C \in \mathbb{N}$,

$$B - C \geq 1.$$

Because both $(a_w, b_w) \in W$ and $(a_w, b) \in W$, we have

$$n \leq a_w C$$

and also

$$a_w \geq rB.$$

Since $r \geq 1$, this gives

$$a_w \geq B.$$

Now consider the pair $(a_w - 1, B)$. We claim it satisfies the product condition:

$$(a_w - 1)B \geq n.$$

Indeed,

$$(a_w - 1)B = a_w B - B = a_w C + a_w(B - C) - B.$$

Because $B - C \geq 1$ and $a_w \geq B$,

$$a_w(B - C) - B \geq a_w - B \geq 0.$$

Therefore

$$(a_w - 1)B \geq a_w C \geq n.$$

So $(a_w - 1, B)$ satisfies $n \leq (a_w - 1)B$.

Now, by definition, $a_w$ is the minimal first coordinate among points in $W$. Since $a_w - 1 < a_w$, the pair $(a_w - 1, B)$ cannot be in $W$. But it already satisfies the product condition, so the only possible failure of membership in $W$ is

$$a_w - 1 < rB.$$

Equivalently,

$$rB \geq a_w - 1.$$

Thus $(a_w - 1, B) \in H$, because

$$n \leq (a_w - 1)B$$

and

$$rB \geq a_w - 1.$$

Since $b_h$ is the minimal second coordinate among points of $H$, and $(a_w - 1, B) \in H$, we get

$$b_h \leq B.$$

Multiplying by $r \geq 1$,

$$rb_h \leq rB.$$

But $(a_w, B) \in W$, so

$$a_w \geq rB.$$

Hence

$$rb_h \leq rB \leq a_w.$$

This contradicts the assumption

$$a_w < rb_h.$$

Therefore no such $b \neq b_w$ exists. So $b_w$ is the unique $b$ such that $(a_w, b) \in W$.