

QED-Nano Proof

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ASSISTANT

▼ Thought for 612 seconds

Proof.

We work with the positive integers $\mathbb{N} = \{1, 2, \dots\}$.

1. Notation and basic facts

- $W = \{(a, b) \in \mathbb{N}^2 \mid n \leq ab, a \geq rb\}$
- $H = \{(a, b) \in \mathbb{N}^2 \mid n \leq ab, rb \geq a\}$
- $(a_w, b_w) = \operatorname{argmin}_{(a,b) \in W} a$ (i.e. $a_w = \min\{a \mid \exists b : (a, b) \in W\}$ and $(a_w, b_w) \in W$).
- $(a_h, b_h) = \operatorname{argmin}_{(a,b) \in H} b$ (i.e. $b_h = \min\{b \mid \exists a : (a, b) \in H\}$ and $(a_h, b_h) \in H$).

We are given

$$a_w < rb_h.$$

2. The integer $L = \lfloor a_w/r \rfloor$

Set

$$L = \lfloor a_w/r \rfloor.$$

From $a_w < rb_h$ we obtain $a_w/r < b_h$; hence

$$L \leq a_w/r < b_h \quad \implies \quad L < b_h. \tag{1}$$

3. $(a_w, L) \in W$

Because $L \leq a_w/r$ we have $a_w \geq rL$.

From $(a_w, b_w) \in W$ we know $a_w b_w \geq n$; therefore

$$b_w \geq \frac{n}{a_w} \implies b_w \geq \lceil n/a_w \rceil.$$

Moreover $b_w \leq a_w/r$ (since $(a_w, b_w) \in W$) and thus $b_w \leq L$ (an integer $\leq a_w/r$ is at most its floor). Consequently

$$L \geq b_w \geq \left\lceil \frac{n}{a_w} \right\rceil. \quad (2)$$

Inequality (2) gives $\frac{n}{a_w} \leq L$, i.e. $a_w L \geq n$. Together with $a_w \geq rL$ this shows

$$(a_w, L) \in W. \quad (3)$$

4. The minimal a for the second coordinate L

For a fixed b define

$$A(b) = \{ a \in \mathbb{N} \mid a \geq rb, ab \geq n \}.$$

If $A(b) \neq \emptyset$ then the smallest element of $A(b)$ is

$$a_{\min}(b) = \max(\lceil rb \rceil, \lceil n/b \rceil).$$

Applying this to $b = L$ and using (3) we have $a_w \in A(L)$; hence $a_{\min}(L) \leq a_w$.

Because a_w is the **global** minimum of the first coordinate in W , every pair $(a, b) \in W$ satisfies $a \geq a_w$. In particular for any b with $A(b) \neq \emptyset$ we must have $a_{\min}(b) \geq a_w$.

Thus for our L (which satisfies $A(L) \neq \emptyset$ by (3))

$$a_{\min}(L) \geq a_w.$$

Combining the two inequalities yields

$$a_{\min}(L) = a_w. \quad (4)$$

5. A consequence of the definition of b_h

Recall that b_h is the smallest integer for which there exists an a with $a \leq rb$ and $ab \geq n$.

Equivalently, for an integer b we have

$$\exists a (a \leq rb, ab \geq n) \iff \lceil n/b \rceil \leq rb.$$

Therefore, for every $b < b_h$ the inequality $\lceil n/b \rceil \leq rb$ fails; i.e.

$$\lceil n/b \rceil > rb. \quad (5)$$

Applying (5) with $b = L$ (note that by (1) $L < b_h$) gives

$$\lceil n/L \rceil > rL. \quad (6)$$

6. Proving $a_w > rL$

Assume, for contradiction, that $a_w = rL$.

From (4) we have $a_w = \max(\lceil rL \rceil, \lceil n/L \rceil)$.

Because $a_w = rL$ must be at least $\lceil n/L \rceil$, (6) would give $\lceil n/L \rceil \leq rL$, contradicting $\lceil n/L \rceil > rL$.

Hence $a_w \neq rL$.

But we already know $a_w \geq rL$ (from $(a_w, L) \in W$). Therefore

$$a_w > rL. \quad (7)$$

7. Determining a_w in terms of L

From (4) and (6) we have

$$a_w = \max(\lceil rL \rceil, \lceil n/L \rceil).$$

Because of (7), $\lceil rL \rceil \leq rL < a_w$; thus $\lceil rL \rceil < a_w$.

Consequently the maximum in (4) must be attained by $\lceil n/L \rceil$ and we obtain

$$a_w = \lceil n/L \rceil. \quad (8)$$

8. An inequality for n

Equation (8) is equivalent to

$$a_w - 1 < \frac{n}{L} \leq a_w \quad \implies \quad L(a_w - 1) < n \leq La_w. \quad (9)$$

9. Showing $b_w = L$

We already know $b_w \leq L$ (from $b_w \leq a_w/r$ and $L = \lfloor a_w/r \rfloor$).

Suppose, contrary to what we want to prove, that $b_w \leq L - 1$.

Then

$$a_w b_w \leq a_w (L - 1) = L a_w - a_w.$$

Because $(a_w, b_w) \in W$ we have $a_w b_w \geq n$; hence

$$L a_w - a_w \geq n. \quad (10)$$

Using (9), $n > L a_w - L$. Together with (10) this yields

$$L a_w - a_w \geq n > L a_w - L \implies -a_w > -L \implies a_w < L.$$

But (7) gives $a_w > rL \geq L$ (since $r \geq 1$). Thus $a_w < L$ and $a_w \geq L$ cannot both hold – a contradiction. Therefore the assumption $b_w \leq L - 1$ is impossible; we must have

$$b_w = L. \quad (11)$$

10. Uniqueness of the second coordinate

From (11) we obtain

$$b_w = \lfloor a_w / r \rfloor.$$

Recall from (2) that $L \geq \lceil n / a_w \rceil$; i.e. $b_w \geq \lceil n / a_w \rceil$.

On the other hand, $(a_w, b_w) \in W$ gives $a_w b_w \geq n$, which implies $b_w \geq \lceil n / a_w \rceil$ (the same inequality). Hence

$$\lceil n / a_w \rceil = b_w = \lfloor a_w / r \rfloor.$$

Consequently the interval

$$\left[\lceil n / a_w \rceil, \lfloor a_w / r \rfloor \right]$$

contains exactly one integer, namely b_w . By definition of W , an integer b satisfies $(a_w, b) \in W$ iff

$$\lceil n / a_w \rceil \leq b \leq \lfloor a_w / r \rfloor.$$

Thus no other integer $b \neq b_w$ can belong to this interval, i.e.

$$\text{there is no } b \neq b_w \text{ with } (a_w, b) \in W.$$

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everything up to 10. is perfect. can you better explain just step 10? how do you reach the conclusion that $\text{ceil}(n/a_w) = b_w$?

ASSISTANT

▼ Thought for 82 seconds

We are asked to elaborate on step 10 of the proof, specifically to explain how we obtain $\lceil n/a_w \rceil = b_w$. We will restate the relevant facts from the previous steps and give a self-contained derivation.

1. Preliminary facts (from steps 1–9)

- $L = \lfloor a_w/r \rfloor$ and we proved $L < b_h$ and $L \geq b_w$; finally we obtained $b_w = L$ (see (11)).

Hence

$$b_w = L = \lfloor a_w/r \rfloor. \quad (\text{A})$$

- From (9) we have

$$L a_w - L < n \leq L a_w.$$

Because $L = b_w$, this reads

$$b_w a_w - b_w < n \leq b_w a_w. \quad (\text{B})$$

- Since $(a_w, b_w) \in W$ we know $a_w \geq r b_w$; together with $r \geq 1$ this gives

$$a_w \geq b_w. \quad (\text{C})$$

- Also $(a_w, b_w) \in W$ implies $a_w b_w \geq n$, i.e.

$$n \leq a_w b_w. \quad (\text{D})$$

2. Deriving $\lceil n/a_w \rceil = b_w$

From (C) we have $a_w \geq b_w$. Therefore

$$a_w(b_w - 1) = a_w b_w - a_w \leq a_w b_w - b_w.$$

(Indeed, subtracting a larger number $-a_w$ instead of $-b_w$ makes the left-hand side smaller.)

Now combine this with (B):

$$a_w(b_w - 1) \leq a_w b_w - b_w < n.$$

Thus

$$a_w(b_w - 1) < n \quad \text{and, by (D),} \quad n \leq a_w b_w.$$

Dividing the first inequality by the positive integer a_w yields

$$b_w - 1 < \frac{n}{a_w} \leq b_w.$$

(The right-hand inequality $\frac{n}{a_w} \leq b_w$ follows from $n \leq a_w b_w$.)

The number $\lceil n/a_w \rceil$ is the smallest integer that is **at least** n/a_w .

Because n/a_w lies strictly between $b_w - 1$ and b_w , the smallest integer $\geq n/a_w$ must be exactly b_w .

Hence

$$\boxed{\lceil n/a_w \rceil = b_w}. \quad (\text{E})$$

3. Consequence for the possible second coordinates

Recall the definition of W :

$$(a, b) \in W \iff n \leq ab \text{ and } a \geq rb.$$

For a fixed $a = a_w$ these conditions become

$$n \leq a_w b \quad \text{and} \quad b \leq \frac{a_w}{r}.$$

Because b is an integer, the second inequality is equivalent to

$$b \leq \lfloor a_w/r \rfloor.$$

From (A) we have $\lfloor a_w/r \rfloor = b_w$.

The first inequality $n \leq a_w b$ is equivalent to

$$b \geq \lceil n/a_w \rceil.$$

Using (E) we obtain $\lceil n/a_w \rceil = b_w$.

Putting everything together:

$$(a_w, b) \in W \iff b \geq b_w \text{ and } b \leq b_w.$$

Thus the only integer b satisfying $(a_w, b) \in W$ is $b = b_w$.

In other words, **there is no other $b \neq b_w$ such that $(a_w, b) \in W$** , which is exactly the statement to be proved.

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